Theor Chim Acta (1986) 70: 119–121

Theoretica Chimica Acta

The incompleteness of Rényi entropies

Werner Hässelbarth

Institut für Quantenchemie, Freie Universität Berlin, Holbeinstr. 48, D-1000 Berlin 45

(Received December 19, 1985, revised January 20/Accepted March 14, 1986)

Since the advent of the notion of mixing character in statistical mechanics, it has been conjectured over and over again that the Rényi entropies provide a mixing isomorphic family, that is: given two probability distributions p and q, the mixing character of q exceeds that of p, $m[q] \succ m[p]$, if and only if $I_\alpha(q) \geq I_\alpha(p)$ for any positive α. This conjecture is disproved by means of counterexamples.

Key words: Rényi entropies — Mixing character — Majorization

The mixing character [1] introduces a pre-order relation to the set $\mathbb{R}_+^n$ of discrete probability distributions $p = (p_1, p_2, \ldots, p_n)$, $p_i \geq 0$, $\Sigma p_i = 1$, as follows: $q \in \mathbb{R}_+^n$ is defined to be more mixed than p, $m[q] \succ m[p]$, if and only if, for any positive integer r, the sum of the r greatest q_i does not exceed the corresponding sum of p_i's. In other words, let p^* and q^* denote the decreasing rearrangements of p and q, respectively. Then

$$m[q] \succ m[p] :\Leftrightarrow \forall (r \geqslant 1): \sum_{i=1}^{r} q_i^* \leqslant \sum_{i=1}^{r} p_i^*.$$

Roughly, increase of mixing character is the mathematical equivalent of increase of statistical disorder, with the equipartition being the most mixed state. While the notion of mixing character was introduced into classical statistics [1], the same ideas were implemented independently into quantum statistics [2], and meanwhile there exists quite some amount of literature on this subject, such as two mathematical monographs [3, 4] on 'majorization' as well as a chapter in a textbook of mathematical physics [5], to mention only some examples. A real valued function $F: \mathbb{R}_+^n \to \mathbb{R}$ is called mixing homomorphic [6] if it provides an order homomorphism, i.e.,

$$m[q] \succ m[p] \Rightarrow F(q) \geq F(p),$$

and a family $\mathscr{F}$ of mixing homomorphic functions is called a mixing-isomorphic one, if

$$m[q] > m[p] \Leftrightarrow \forall (F \in \mathscr{F})\colon F(q) \geq F(p).$$

The Rényi entropies I_α [7] are characterized by a real positive parameter α. Their definition reads as follows:

$$I_\alpha(p) = \frac{1}{1-\alpha} \log \sum_{i=1}^{n} p_i^\alpha \quad \text{for } \alpha \neq 1,$$

and

$$I_1(p) = \lim_{\alpha \to 1} I_\alpha(p) = -\sum_{i=1}^{n} p_i \log p_i,$$

so I_1 coincides with the Shannon-Boltzmann entropy. For convenience we also include the limits

$$I_0(p) = \lim_{\alpha \to 0} I_\alpha(p) = \log (\text{no. of nonzero } p_i),$$

$$I_\infty(p) = \lim_{\alpha \to \infty} I_\alpha(p) = -\log \max_i p_i.$$

The Rényi entropies are readily shown to be mixing homomorphic; however, they do not constitute a mixing-isomorphic family. We prove that by specifying $q, p \in \mathbb{R}_+^n$ such that $I_\alpha(q) \geq I_\alpha(p)$ for any $\alpha \geq 0$, while q and p are incomparable, i.e., neither $m[q] > m[p]$ nor $m[q] < m[p]$ holds.

This negative result, to be demonstrated below, is by now quite a few years old, since the present author hesitated to publish it. It originates in the Elmau Conference on Irreversible Thermodynamics (1975), where the conjecture about the completeness of the Rényi entropies (i.e., that they do form a mixing isomorphic family) was raised. Since then the present author met this very same conjecture over and over again [8] – the family of Rényi entropies being distinguished by sharing the property of additivity with the Shannon-Boltzmann entropy. So this modest result may be worthwhile publishing, in the end. It should be emphasized, however, that the incompleteness of the Rényi entropies is certainly known among some measure of the mathematical physics community, compare, e.g., [9].

Counterexample. We skip normalization to unity and choose

$$q = (8, 8, 8, 8, 1, 1, 1, 1),$$

$$p = (16, 4, 4, 4, 4, 4, 0, 0).$$

The q_i and p_i are already in decreasing order, and their partial sums read

$$8, 16, 24, 32, 33, 34, 35, 36$$

$$16, 20, 24, 28, 32, 36, 36, 36$$

so neither $m[q] > m[p]$ nor $m[q] < m[p]$ holds. $I_\alpha(p)$, for any fixed p, is a monotonously decreasing function of α. We explicitely compute, now for q and p normalized to unity,

$$I_0(q) = \log 8, \qquad I_0(p) = \log 6,$$
$$I_1(q) = \log 36 - \tfrac{24}{9} \log 2, \qquad I_1(p) = \log 36 - \tfrac{26}{9} \log 2,$$
$$I_\infty(q) = \log 36 - 3 \log 2, \qquad I_\infty(p) = \log 36 - 4 \log 2.$$

So, for these three values of α, $I_\alpha(q) > I_\alpha(p)$. In order to show that this holds for any $0 < \alpha < \infty$, $\alpha \neq 1$, it is sufficient, because of continuity, to check that $I_\alpha(q) = I_\alpha(p)$ cannot happen for any such α, i.e. we have to demonstrate that there is no positive $\alpha \neq 1$ such that

$$4 \cdot 8^\alpha + 4 = 16^\alpha + 5 \cdot 4^\alpha$$

Introducing $x = 2^\alpha$, we have to solve

$$x^4 - 4x^3 + 5x^2 - 4 = 0.$$

By factorizing this polynomial according to

$$x^4 - 4x^3 + 5x^2 - 4 = (x-2)(x(x-1)^2 + 2),$$

it is found to have one positive zero, $x = 2$, corresponding to $\alpha = 1$, of course, but there is no other one since $x(x-1)^2 \geq 0$ for positive x. So there is no $0 < \alpha < \infty$, $\alpha \neq 1$ such that $I_\alpha(q) = I_\alpha(p)$. Thus our counterexample is properly established.

As a final remark, there are larger counterexamples that disprove the conjecture, that, in terms of partition diagrams, inclusion of the Rényi entropies referring to the columns, besides those referring to the rows, provides a mixing isomorphic family.

$$q' = (8, 8, 8, 8, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$
$$p' = (16, 4, 4, 4, 4, 4, 4, 4, 0, 0, 0, 0, 0, 0, 0, 0)$$

q' and p' are incomparable and mutual associates at the same time, and $I_\alpha(q') > I_\alpha(p')$ for all $\alpha \geq 0$.

Acknowledgement. The author wishes to acknowledge encouragement by Prof. J. Brickmann to publish this note.

References

1. Ruch E (1975) Theor Chim Acta 38:167
2. Uhlmann A (1971) Wiss Z KMU, Leipzig 20:633
3. Marshall A W, Olkin I (1979) Inequalities: Theory of majorization and its applications. Academic Press, New York
4. Ando T (1982) Majorization, doubly stochastic matrices, and comparison of eigenvalues. Hokkaido University, Sapporo
5. Thirring W (1980) Lehrbuch der Mathematischen Physik, vol 4. Springer, Wien
6. Ruch E, Mead A (1976) Theor Chim Acta 41:95
7. Rényi A (1973) Wahrscheinlichkeitstheorie. Deutscher Verlag der Wissenshaften, Berlin
8. Primas H: personal communication, Brickmann J: personal communication, to cite only two examples
9. Wehrl A (1978) Rev Mod Phys 50:221